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CRT-749

SOLVING THE SINGLE ARTERY TRAFFIC SIGNAL SYNCHRONIZATION PROBLEM WITH BENDERS DECOMPOSITION*

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* This research has been supported by a grant from the NSERC.

Centre de recherche sur les transports — Publication #749

January 1991

Dépôt légal – 1er trimestre 1991

Abstract

The green band method has been used by traffic engineers since the 1950's because of its simplicity and its graphical representation. Little[1966] developed the green band maximization problem formulation and it was implemented in the MAXBAND computer program distributed by the U.S. Department of Transportation. This paper describes this formulation and shows how to decompose the problem according to Benders' method. The subproblem produced by this decomposition has a network flow-like structure and we develop a special algorithm that solves it in $O(n^2)$ time, where n is the number of signals on the artery. The CPU times required by this approach show that it is about 3 to 10 times faster than the MAXBAND approach with arteries ranging in size from 5 to 20 signals.

Key words: Traffic signals, Signal synchronization, Green band, MAXBAND, Integer programming, Benders decomposition, Network flow.

Résumé

La méthode de la bande verte est utilisée depuis les années 50 par les ingénieurs chargés de la circulation parce qu'elle est simple et elle peut être présentée graphiquement. Little[1966] a développé le modèle de maximisation de la bande verte qui est implanté dans le logiciel MAXBAND distribué par le Département des transports du gouvernement américain. Cet article décrit le problème et montre comment le décomposer selon la méthode de Benders. Le sous-probème produit par cette décomposition possède une structure de flot dans un réseau et nous présentons un algorithme spécialisé qui peut le résoudre en temps $O(n^2)$, où n est le nombre de feux de circulation sur l'artère. Les temps d'ordinateur montrent que cette méthode est de 3 à 10 fois plus rapide que la méthode de MAXBAND avec des artères de 5 à 20 feux de circulation.

Mots clés: Feux de circulation, Synchronisation des feux, Bande verte, MAXBAND, Programmation en nombres entiers, Décomposition de Benders, Flot dans un réseau.

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Synchronizing traffic signal on an artery is essential to avoid congestion and to provide a smooth traffic flow. The benefits of a good synchronization include, among others, better fuel consumption and less driver aggravation.

The most popular method used by traffic engineers is called the green band or progression method. This method's objective is to maximize the width of a time window in which vehicles can travel from one end of an artery to the other without being stopped by a red signal. Even though this method ignores important aspects of reality, like traffic density, pedestrian interference, bus lanes, etc., it is generally accepted as giving good results.

Solution methods have ranged from scale models with threads and strips of paper with red markings (Petterman [1947]) to heuristics assuming a perfect geometry or equal green times (Bowers[1947] and Matson et al.[1955]), to an exact algorithm using a mixed-integer mathematical programming formulation (Little[1966]).

The gasoline crises of 1973 and 1979 have raised the social concern of using energy efficiently, and the U.S. Department of Transportation is encouraging cities to help conserve fuel by distributing a free computer program called MAXBAND developed by Little et al.[1981] and revised by Messer et al.[1986]. It believes that the fuel saved by such measures is significant.

In this paper, we will first describe the mixed-integer linear programming formulation adapted from Little[1966]. Then, we will show, in Section 2, how to apply the Benders decomposition algorithm to this problem. In Section 3, we develop an algorithm which solves the Benders subproblem by taking advantage of its network flow-like structure. In Section 4, we show how to extract, from the subproblem's solution, the dual variables needed to build a new constraint for the Master problem. Finally, in Section 5, we present some computational results to compare the Benders decomposition approach to the branch and bound used in MAXBAND.

1 PROBLEM FORMULATION

The green band problem formulation is based on the space-time diagram (Figure 1), a graphic representation of an artery used to represent the signal settings and their impact on traffic. The vertical axis represents space, on which we draw a scaled version of the artery, and the horizontal axis represents time, on which we draw the phases of the signals. Signals will be numbered consecutively from the bottom to the top of the y-axis. We say that cars travelling up are going in direction 1 and cars traveling down in direction 2.

Any point on the diagram can represent a car's position on the artery at a given moment in time, but it does not show its speed or its direction. A segment between two points representing the same car at different moments in time is called a trajectory: the magnitude of its slope is its speed and the sign of its slope indicates its direction: segment AB of Figure 1 represents a car going in direction 2 at a speed of 5 meters per second.

Figure 2 shows the trajectory of 5 cars, all travelling at a constant speed of 7.5 m/s, except for car C_1 which has to stop during the red phase of signal S_3. We can easily see that any car arriving before C_2 and after C_3 will have to stop at least once on its way to S_5. The time window during which cars can go from S_1 to S_5 without having to stop is called the green band. There can be a green band in each direction and our objective is to maximize

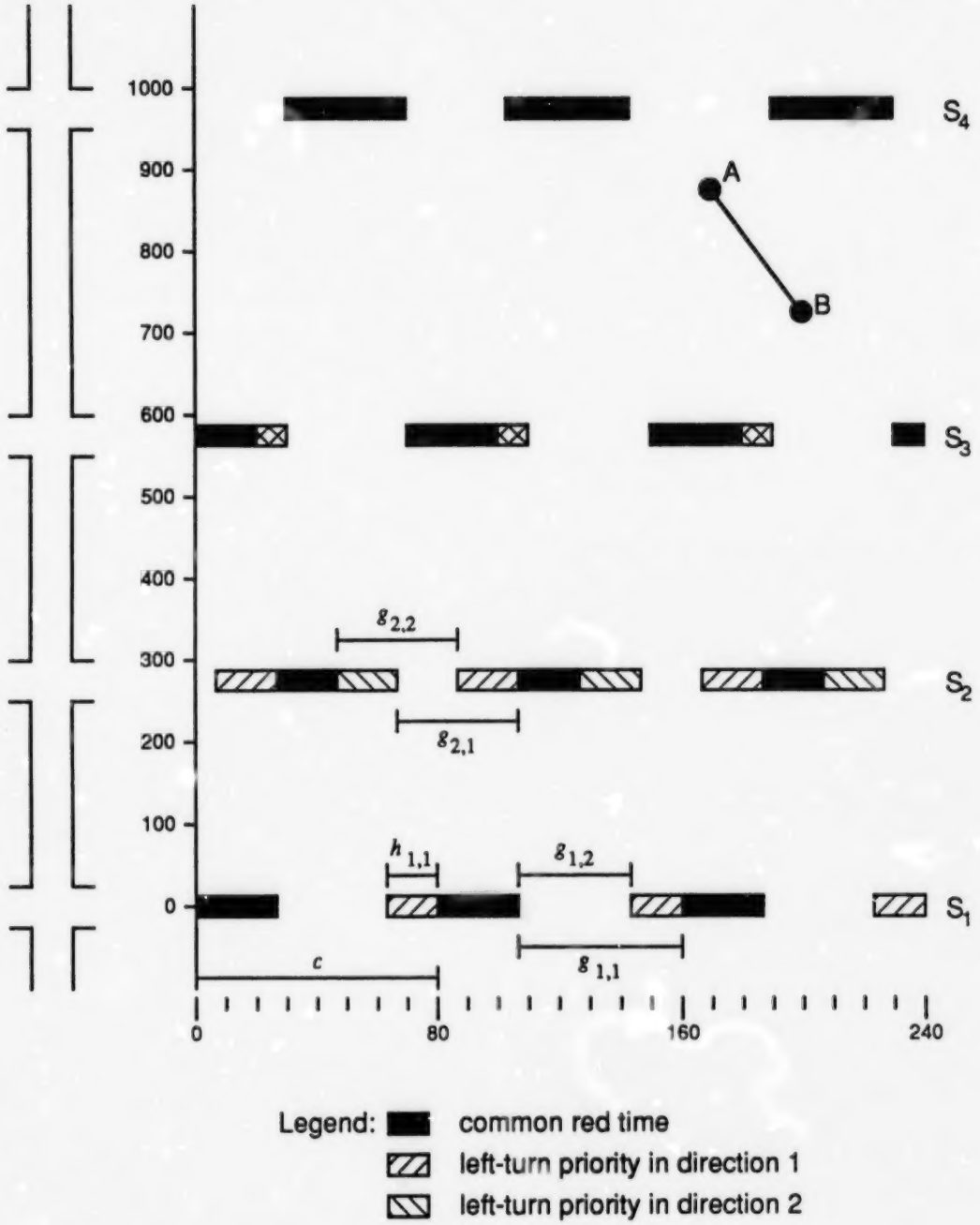


Figure 1: Space–Time Diagram

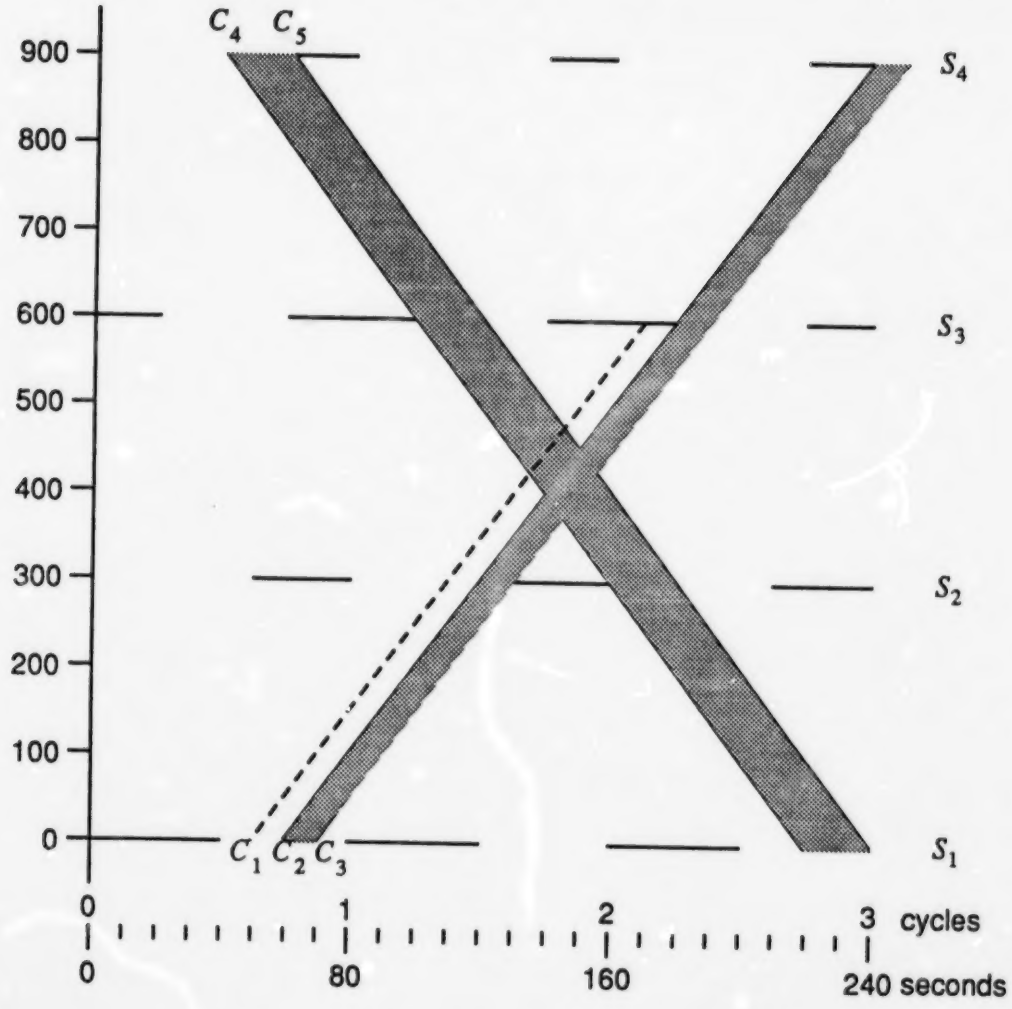


Figure 2: Trajectories

the smaller of the two.

We will need the following constants in our problem formulation:

n	number of signals,
$g_{i,d}$	duration (cycles) of the green phase of S_i in direction d,
$l_{i,d}$	duration (cycles) of the left-turn priority at S_i in direction d,
$d_{i,d}$	distance (meters) from S_i to the next signal in direction d (note that $d_{n,1}$ and $d_{1,2}$ do not really exist: for convenience, we will let them equal 0,)
$v^+_{i,d}$, $v^-_{i,d}$	upper and lower bound of the target speed (meters/second) when leaving signal S_i in direction d,
c^+, c^-	upper and lower bound of the cycle duration (seconds/cycle).

We determine the optimal values of the following variables:

b	duration (cycles) of the overall green band,
b_d	duration (cycles) of the green band in direction d,
$w_{i,d}$	waiting time (cycles) at S_i for the green band going in direction d,
m_i	U-turn integer (cycles),
$h_{i,d}$	left-turn position indicator (no units): it equals 1 if the left-turn priority $l_{i,d}$ occurs *after* the red phase and 0 if it occurs *before*,
$t_{i,d}$	travel time (cycles) from S_i to the next signal in direction d, and
s	cycle frequency (cycles/second).

The mixed-integer linear programming formulation of the traffic signal synchronization problem for a single artery with variable speed and variable cycle time is:

$$\text{Max } b + \alpha(b_1 + b_2) \qquad \text{(SAVS)}$$

subject to

$$b \le b_d \qquad d = 1,2, \tag{1}$$

$$w_{i,d} + b_d \le g_{i,d} \qquad i = 1 \ldots n, d = 1,2, \tag{2}$$

$$(w_{i,1} + w_{i+1,2}) - (w_{i,2} + w_{i+1,1}) + (t_{i,1} + t_{i+1,2}) + (l_{i,2}h_{i,2} + l_{i+1,1}h_{i+1,1}) - (l_{i,1}h_{i,1} + l_{i+1,2}h_{i+1,2}) = m_i \qquad i = 1 \ldots n-1, \tag{3}$$

$$s \times d_{i,d}/v^+_{i,d} \le t_{i,d} \le s \times d_{i,d}/v^-_{i,d} \qquad i = 1 \ldots n, d = 1,2, \tag{4}$$

$$(1/c^+) \le s \le (1/c^-) \tag{5}$$

$$w_{i,d} \ge 0 \qquad i = 1 \ldots n, d = 1,2,$$

$$h_{i,d} = 0 \text{ or } 1 \qquad i = 1 \ldots n, d = 1,2,$$

$$m_i \text{ integer} \qquad i = 1 \ldots n-1,$$

$$b, b_1 \text{ and } b_2 \text{ unrestricted.}$$

Constraint (1) forces the overall bandwidth b to equal the smaller of the bandwidths in each direction. Parameter α is an arbitrarily small constant that will force the maximization of both bandwidths when they are unequal. The objective function and constraint (1) will try to make variables b, b_1 and b_2 equal whenever possible, in which case the objective function becomes $(1 + 2\alpha)b$ which, in turn, is equivalent to maximizing b. But when the optimal bandwidths are different in each direction, α will have no impact on the optimal bandwidth but it will choose, among the multiple solutions that give that bandwidth, the solution that

	Maneuver description	Elapsed time
0	The car starts out at the end of the red phase of S_i,	
1	it waits until the end of the left-turn priority time in the opposite direction, if there is some,	$l_{i,2}h_{i,2}$
2	it waits for the green band in direction 1,	$w_{i,1}$
3	it travels to S_{i+1},	$t_{i,1}$
4	it waits until the end of the red phase at S_{i+1}, during which time it makes a U-turn,	$1 - w_{i+1,1} - l_{i+1,2}h_{i,2}$
5	now, it is facing direction 2 and it will repeat the same maneuvers: it waits until the end of the left-turn priority time in the opposite direction,	$l_{i+1,1}h_{i+1,1}$
6	it waits for the green band in direction 2,	$w_{i+1,2}$
7	it travels to S_i, and	$t_{i+1,2}$
8	finally it waits until the end of the red phase at S_i,	$1 - w_{i,2} - l_{i,1}h_{i,1}$

Table 1: U-turn maneuvers

maximizes the bandwidth in the other direction. Variables b, b_1 and b_2 are unrestricted because it will ensure that problem (SAVS) always has a feasible solution which makes the subproblem algorithm much simpler. In practice, though, a negative optimal bandwidth is equivalent to an infeasible problem.

Constraint (2) states that, for each signal in each direction, the time between the end of the red phase and the start of the green band plus the green band itself must not be greater than the available green time.

Constraint (3) is called the U-turn constraint. It is the only constraint that links the signals together and it represents a car making a series of maneuvers between two signals (see Table 1 and Figure 3.) The maneuvers start at a pre-determined moment in time (i.e. the end of the red phase of S_i) and end at a similar moment later in time (i.e. the end of the red phase of S_i again). Thus, the sum of the elapsed time (measured in cycles) of all the maneuvers must equal an integer number of cycles.

Constraint (4) is the travel time bounds and constraint (5) is the cycle frequency bounds. We use the cycle frequency instead of the cycle time only to make constraint (4) linear.

2 PROBLEM DECOMPOSITION

Problem (SAVS) has $5n + 2$ constraints, excluding nonnegativity, $4n + 2$ continuous variables, and from $n - 1$ to $3n - 1$ integer variables, depending on the number of left-turn priorities we consider. It can be solved by any general mixed-integer linear programming code: MAXBAND (Little[1981]) uses the branch and bound code of Land and Powell[1973].

We wish to propose another solution method that takes advantage of the special structure of this problem. The method is an implementation of the Benders decomposition algorithm. It consists of proposing a feasible solution and verifying its optimality: if the proposed solution is provably optimal, then the problem is solved, and if it is not then we obtain some information that permits us to make another proposal.

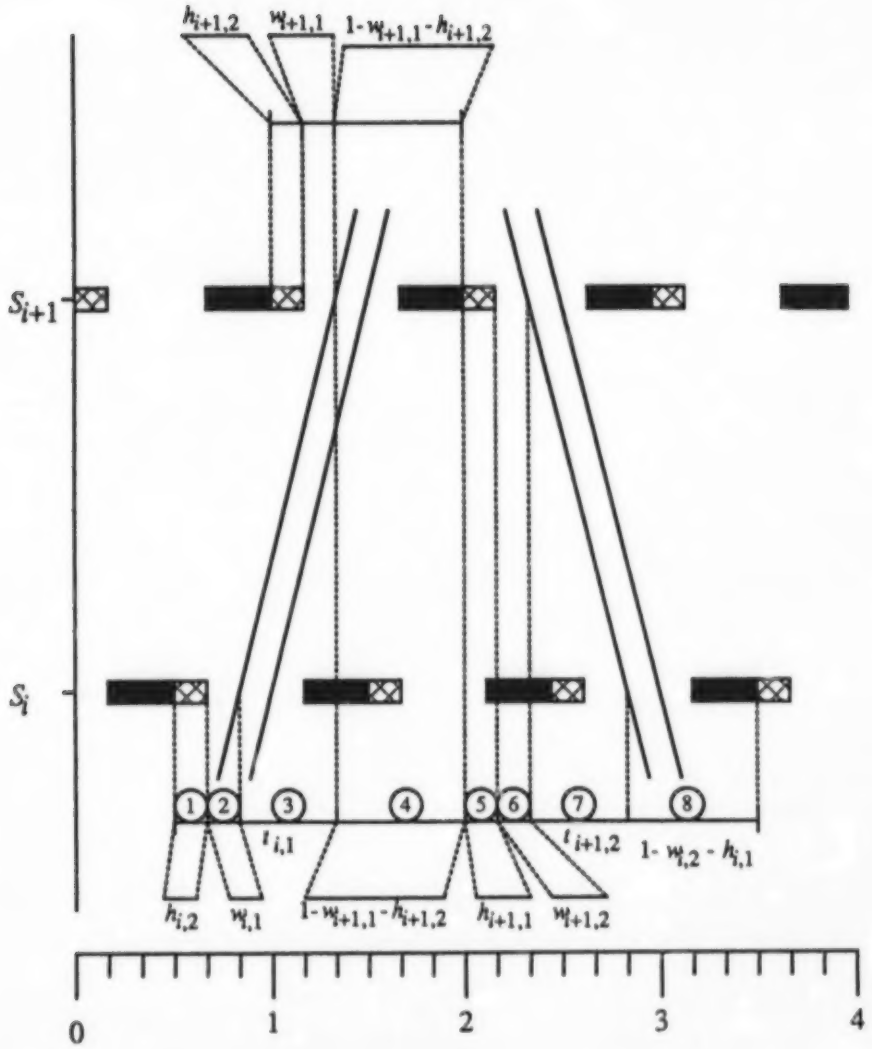


Figure 3: U–turn Constraint

We will let the Master problem take care of the complicating variables: the m_i's, the $h_{i,d}$'s and s. The subproblem then becomes:

$$b(m,h,s) = \text{Max } b + \alpha(b_1 + b_2) \qquad \text{(SP)}$$

subject to

$$\begin{aligned}
& b \le b_d \qquad d = 1,2 \\
& w_{i,d} + b_d \le g_{i,d} \qquad i = 1 \ldots n, d = 1,2 \\
& (w_{i,1} + w_{i+1,2}) - (w_{i,2} + w_{i+1,1}) + (t_{i,1} + t_{i+1,2}) \\
& \qquad = m_i - (l_{i,2}h_{i,2} + l_{i+1,1}h_{i+1,1}) + (l_{i,1}h_{i,1} + l_{i+1,2}h_{i+1,2}) \qquad i = 1 \ldots n-1 \qquad (6) \\
& s \times d_{i,d}/v^+_{i,d} \le t_{i,d} \le s \times d_{i,d}/v^-_{i,d} \qquad i = 1 \ldots n, d = 1,2, \\
& w_{i,d} \ge 0 \qquad i = 1 \ldots n, d = 1,2, \\
& b, b_1 \text{ and } b_2 \text{ unrestricted.}
\end{aligned}$$

That is, $b(m,h,s)$ is the maximum green band for some fixed U-turn times, left-turn time positions and cycle frequency.

To simplify the notation, we will let

$$U_i(m,h) = m_i - (l_{i,2}h_{i,2} + l_{i+1,1}h_{i+1,1}) + (l_{i,1}h_{i,1} + l_{i+1,2}h_{i+1,2}) \qquad i = 1 \ldots n-1,$$

$$t^-_{i,d}(s) = s \times d_{i,d}/v^+_{i,d} \qquad i = 1 \ldots n, d = 1,2$$

$$t^+_{i,d}(s) = s \times d_{i,d}/v^-_{i,d} \qquad i = 1 \ldots n, d = 1,2$$

Letting e_d, $q_{i,d}$, p_i, $u^-_{i,d}$ and $u^+_{i,d}$ be the dual variables associated with, respectively, the bandwidth constraints, the green time constraints, the U-turn constraints and the travel time bounds, we can write the dual problem of (SP) as follows:

$$b(m,h,s) = \text{ Min } \sum g_{i,d}q_{i,d} + \sum U_i(m,h)p_i + \sum (t^+_{i,d}(s)u^+_{i,d} - t^-_{i,d}(s)u^-_{i,d}) \qquad \text{(DSP)}$$

subject to

$$\begin{aligned}
& e_1 + e_2 = 1 && (7) \\
& \sum q_{i,d} - e_d = \alpha \qquad d = 1,2, && (8) \\
& q_{i,1} + (p_i - p_{i-1}) \ge 0 \qquad i = 1 \ldots n, && (9) \\
& q_{i,2} - (p_i - p_{i-1}) \ge 0 \qquad i = 1 \ldots n, && (10) \\
& u^-_{i,1} - u^-_{i,1} + p_i \ge 0 \qquad i = 1 \ldots n-1, && (11) \\
& u^-_{i+1,2} - u^-_{i+1,2} + p_i \ge 0 \qquad i = 1 \ldots n-1, && (12) \\
& q_{i,d}, u^+_{i,d}, u^-_{i,d} \ge 0 \qquad i = 1 \ldots n, d = 1,2, \\
& p_i \text{ unrestricted} \qquad i = 1 \ldots n-1.
\end{aligned}$$

The Master problem will then be:

$$z^K = \text{Max } z \qquad \text{(MP)}$$

subject to

$$z \leq \sum g_{i,d} q_{i,d}^k + \sum U_i(m,h) p_i^k + \sum (t_{i,d}^+(s) u_{i,d}^{k+} - t_{i,d}^-(s) u_{i,d}^{k-}) \qquad k = 1 \ldots K$$
$$(1/c^+) \leq s \leq (1/c^-)$$
$$h_{i,d} = 0 \text{ or } 1 \qquad i = 1 \ldots n, d = 1,2,$$
$$m_i \text{ integer} \qquad i = 1 \ldots n-1.$$

Problem (MP) has the same integer variables as (SAVS), two continuous variable, and K constraints, and it can be solved by any available mixed-integer linear programming code. It is smaller than (SAVS), and this is where we expect to save on computation time; if K remains small enough, solving (MP) K times will be faster than solving (SAVS) once.

Subproblem (SP) can be solved with any available linear programming code, but it is possible to take advantage of its special structure to find an optimal solution more efficiently. The next section will consider this possibility.

In the following sections, we will drop the function references to simplify the notation of the subproblem: we simply use U_i, $t_{i,d}^-$ and $t_{i,d}^+$ with the understanding that they are functions of m, h and/or s.

3 SUBPROBLEM SOLUTION

3.1 Critical and Binding Signals

For any given solution (w,t), we can compute its corresponding bandwidths as follows:

$$b_d = \min_{i=1,\ldots n} (g_{i,d} - w_{i,d}) \qquad d = 1,2$$

and

$$b = \min(b_1, b_2).$$

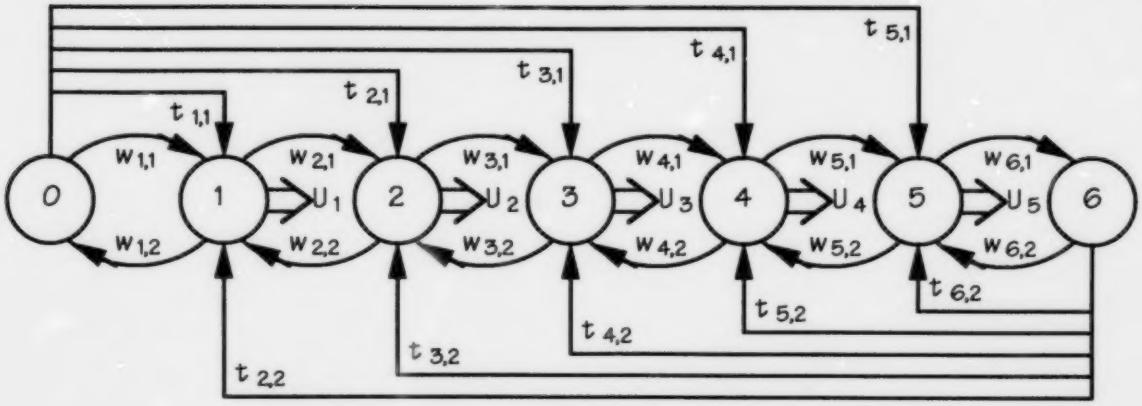
We will distinguish two types of signals. We say that a signal is *binding*, in direction d, if $g_{i,d} = b_d$ and $w_{i,d} = 0$. For a given solution, there may be zero, one or two binding signals. We will use the subscript j_d to denote binding signals.

We say that a signal is *critical*, in direction d, if $g_{i,d} - b_d = w_{i,d}$ and $w_{i,d} = 0$. We will use the subscript i_d to denote critical signals.

3.2 Matrix Representation

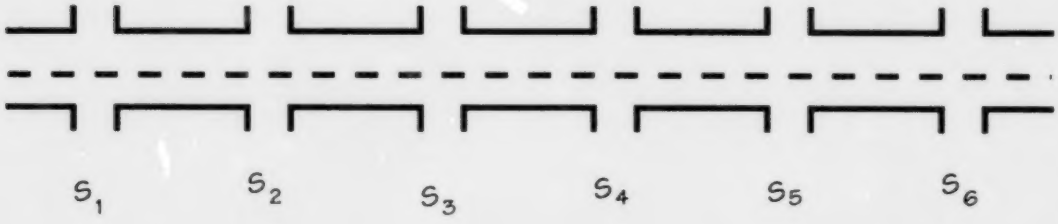
We can view problem (SP) as a network flow problem (see Figure 4) with arcs $(i-1,i)$ carrying flow $w_{i,1}$, arcs $(i,i-1)$ carrying flow $w_{i,2}$, arcs $(0,i)$ carrying flow $t_{i,1}$ and arcs (n,i) carrying flow $t_{i+1,2}$. The U_i's can be viewed as node requirements and constraint (6) becomes a flow balance constraint. Note that nodes 0 and n have no corresponding U_i and, therefore, they can be considered as unlimited sources and/or supplies.

We introduce, in this section, the matrix representation of the network flow representation of constraint (6) of problem (SP), along with definitions that will be helpful in the construction of algorithm to solve the subproblem.



Network Flow Representation

(a)



Corresponding Artery Diagram

(b)

Figure 4: Network flow view of subproblem

Column vector $\mathbf{w}$ will represent the problem variables $w_{i,d}$'s and $t_{i,d}$'s, matrix $\mathbf{A}$ will be the node-arc incidence matrix representing constraints (6), and column vector $\mathbf{u}$ will represent the problem constants U_i's. Vector $\mathbf{w}$ has $4n-2$ rows (the $2n$ $w_{i,d}$'s and the $2(n-1)$ $t_{i,d}$'s), vector $\mathbf{u}$ has $n-1$ rows and matrix $\mathbf{A}$ has $n-1$ rows and $4n-2$ columns. We can then write all the constraints (6) as follows:

$$\mathbf{Aw} = \mathbf{u}. \tag{13}$$

We say that vector $\mathbf{w}$ is a feasible solution if it satisfies (13) and if each of its elements satisfies its bounds, i.e. the $w_{i,d}$'s are nonnegative and the $t_{i,d}$'s are between their lower and upper bounds.

We will call a column vector δ, with the same number of elements as $\mathbf{w}$, a circuit if $\mathbf{A}\delta = \mathbf{0}$, where $\mathbf{0}$ is a column vector with as many elements as $\mathbf{u}$, all of whose components are equal to zero. Furthermore, such a circuit will be called *feasible with respect to a feasible solution* $\mathbf{w}$ if $\tilde{\mathbf{w}} = \mathbf{w} + \delta$ is also a feasible solution. We will call a feasible circuit δ an *augmentation circuit* if $\tilde{b} + \alpha(\tilde{b}_1 + \tilde{b}_2) > b + \alpha(b_1 + b_2)$, where $\tilde{b}$, $\tilde{b}_1$ and $\tilde{b}_2$ are the bandwidths associated with the feasible solution $\tilde{\mathbf{w}}$ and b, b_1 and b_2 with $\mathbf{w}$.

3.3 Optimal Solution Properties

Subproblem (SP) has some properties that will help us characterize an optimal solution. These properties can all be interpreted in terms of flows with respect to the node-arc incidence matrix $\mathbf{A}$.

Property 1 *A solution* $\mathbf{w}^*$ *is optimal if and only if there is no augmentation circuit with respect to it.*

Proof:

only if: there is no augmentation circuit if $\mathbf{w}^*$ is optimal. Assume there exists an augmentation circuit δ. Then, $\tilde{\mathbf{w}} = \mathbf{w}^* + \delta$ is a feasible solution with $\tilde{b} + \alpha(\tilde{b}_1 + \tilde{b}_2) > b^* + \alpha(b_1^* + b_2^*)$ which contradicts the fact the $\mathbf{w}^*$ is optimal.

if: $\mathbf{w}^*$ is optimal if there is no augmentation circuit. Assume that $\mathbf{w}^*$ is not optimal, but that $\tilde{\mathbf{w}}$ is. Let $\delta = \tilde{\mathbf{w}} - \mathbf{w}^*$. Then, δ is an augmentation circuit because $\mathbf{A}\delta = \mathbf{A}(\tilde{\mathbf{w}} - \mathbf{w}^*) = \mathbf{A}\tilde{\mathbf{w}} - \mathbf{A}\mathbf{w}^* = \mathbf{u} - \mathbf{u} = \mathbf{0}$ and $\tilde{b} + \alpha(\tilde{b}_1 + \tilde{b}_2) > b^* + \alpha(b_1^* + b_2^*)$ which contradicts the fact that there were no augmentation paths with respect to $\mathbf{w}^*$. ∎

Property 2 *Subproblem (SP) has an optimal solution* $\mathbf{w}^*$*, where* $w_{i,1}^* w_{i,2}^* = 0$ *(i.e. at least one of* $w_{i,1}^*$ *or* $w_{i,2}^*$ *equals zero) for* $i = 1$ *to* n*.*

Proof:

Let $\tilde{\mathbf{w}}$ be any optimal solution. We can transform it into a solution $\mathbf{w}^*$ satisfying property 2 with the following procedure:

- let $\delta_i = \min(\tilde{w}_{i,1}, \tilde{w}_{i,2})$ for $i = 1$ to n,

- let $w^*_{i,d} = \tilde{w}_{i,d} - \delta_i$, for $i = 1$ to n and $d = 1$ and 2,

- let $t^*_{i,d} = \tilde{t}_{i,d}$, for $i = 1$ to n and $d = 1$ and 2.

The new solution is feasible since constraint (6) is satisfied:

$$\begin{aligned}
&(w^*_{i,1} + w^*_{i+1,2}) - (w^*_{i,2} + w^*_{i+1,1}) + (t^*_{i,1} + t^*_{i+1,2}) \\
&= (\tilde{w}_{i,1} - \delta_i + \tilde{w}_{i+1,2} - \delta_{i+1}) - (\tilde{w}_{i,2} - \delta_i + \tilde{w}_{i+1,1} - \delta_{i+1}) + (\tilde{t}_{i,1} + \tilde{t}_{i+1,2}) \\
&= (\tilde{w}_{i,1} + \tilde{w}_{i+1,2}) - (\tilde{w}_{i,2} + \tilde{w}_{i+1,1}) + (\tilde{t}_{i,1} + \tilde{t}_{i+1,2}) \\
&= U_i
\end{aligned}$$

The new solution is also optimal since all the δ_i's are nonnegative, which implies that $w^*_{i,d} \leq \tilde{w}_{i,d}$ and, consequently, $b^* \geq \tilde{b}$. ∎

Property 3 *Let* $\mathbf{w}^*$ *be an optimal solution with two critical signals,* i_1 *and* i_2*:*
if $i_1 < i_2$ *then* $t^*_{i,1} = t^+_{i,1}$ *and* $t^*_{i+1,2} = t^+_{i+1,2}$ *for* $i = i_1$ *to* $i_2 - 1$,
if $i_1 > i_2$ *then* $t^*_{i,1} = t^-_{i,1}$ *and* $t^*_{i+1,2} = t^-_{i+1,2}$ *for* $i = i_2$ *to* $i_1 - 1$.

Proof:

We will prove this result only for the case $i_1 < i_2$, the other case being similar. Let $\mathbf{w}$ be a solution, as illustrated in Figure 5(a), for which there exists a signal i', $i_1 \leq i' < i_2$, with $t_{i',1} < t^+_{i',1}$: we will show how to construct an augmentation circuit with respect to w which will increase the value of b^*_1.

- Build a path on which we can send $-\delta$ units from node 0 to node i' using the $w_{i,d}$'s as follows (see Figure 5(b)):
  - if $w_{i,1} > 0$, decrease $w_{i,1}$ by $-\delta$
  - otherwise, increase $w_{i,2}$ by δ.

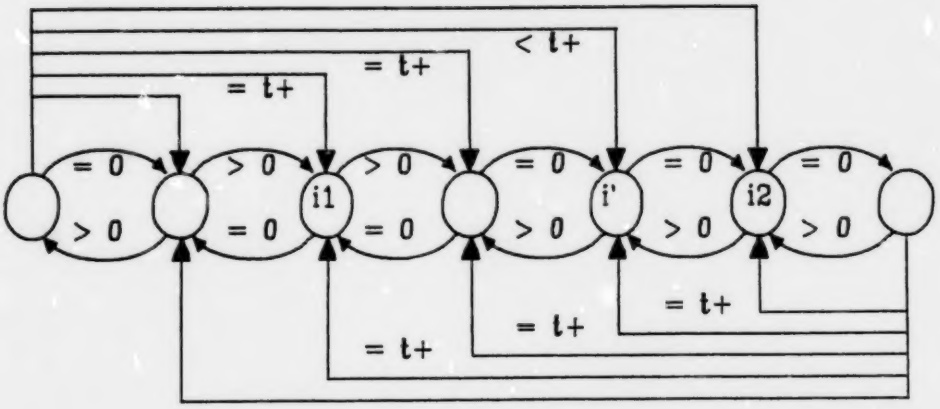
    Since none of the $w_{i,2}$ used are critical in direction 2, for some $\delta > 0$ small enough the value of b_2 will not change during this step.
- Complete the circuit by increasing $t^+_{i',1}$ by δ.

At this point, we have increased the value of b^*_1 by δ, since $w_{i_1,1}$ decreased by δ, proving that our original solution $\mathbf{w}$ was not optimal. ∎

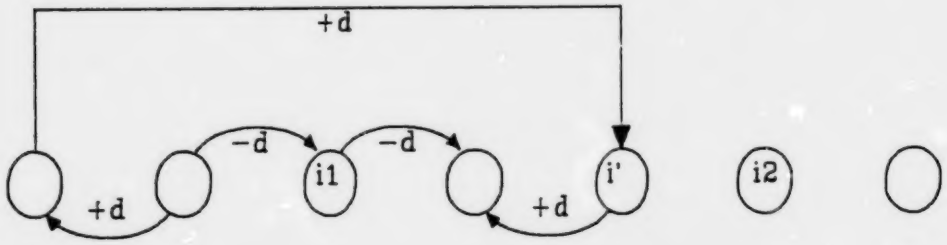
Property 4 *Let* $\mathbf{w}^*$ *be an optimal solution with no binding signal, then* $b^*_1 = b^*_2$.

Proof:

If $b^*_1 \neq b^*_2$ then we could build an augmentation circuit with which we can balance the two bandwidths and, therefore, obtain a better solution. As illustrated in Figure 6(a), assume that $b^*_1 < b^*_2$ and $i_1 < i_2$, then (see Figure 6(b)) the circuit (S, $+w_{1,2}, \ldots, +w_{i_1-1,2}, -w_{i_1,1}, +w_{i_1+1,2}, \ldots, +w_{n,2}, S$), with flow $\delta = (b^*_2 - b^*_1)/2$, will give us a new feasible solution with equal bandwidths and contradicts the fact that $\mathbf{w}^*$ was optimal. ∎

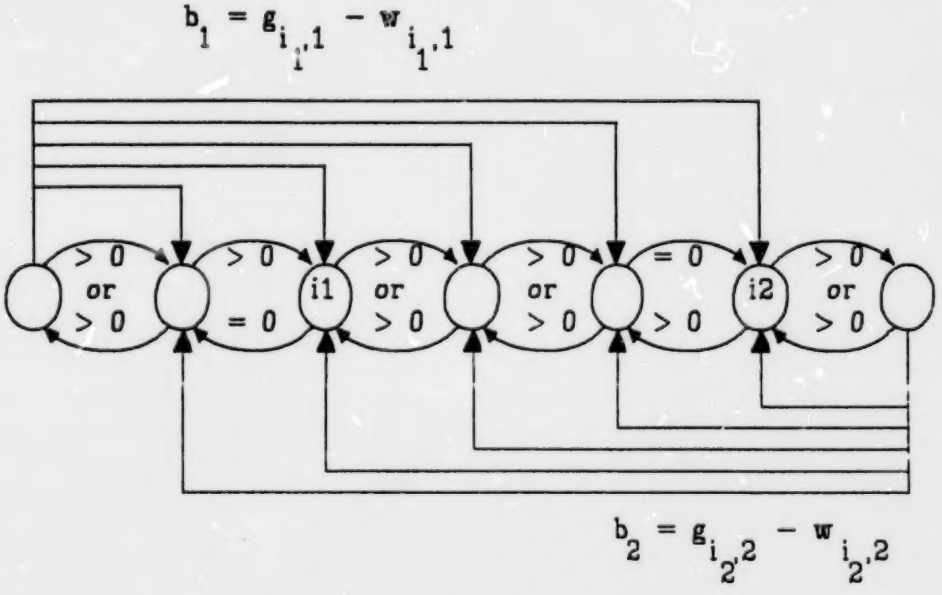


(a)

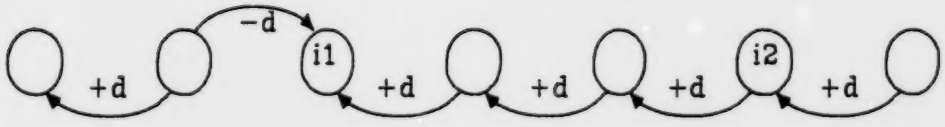


(b)

Figure 5: Proof of Property 3



(a)



(b)

Figure 6: Proof of Property 4

Property 5 *Let* $\mathbf{w}^*$ *be an optimal solution with one binding signal,* j_1 *or* j_2*:*
- *if* j_1 *is the binding signal, then* $b^* = b_1^* \leq b_2^*$,
- *if* j_2 *is the binding signal, then* $b^* = b_2^* \leq b_1^*$.

Proof:

We will prove this result only for the case in which j_1 is the binding signal, as illustrated in Figure 7(a). If $b^* \neq b_1^*$ then we must have $b^* = b_2^* < b_1^*$. We could use the circuit $(S, +w_{1,1}, \ldots, +w_{i_2-1,1}, -w_{i_2,2}, +w_{i_2+1,1}, \ldots, +w_{n,1}, S)$ with $\delta > 0$ to increase b_2 and get a better solution, which contradicts the fact that $\mathbf{w}^*$ was optimal. ∎

Property 6 *Let* $\mathbf{w}^*$ *be an optimal solution with one binding signal,* j_1 *or* j_2*, then it must have two critical signals.*

Proof:

Let $\mathbf{w}^*$ be an optimal solution with one binding signal, j_1. According to the definition of a critical signal, there must be at least one signal i_2 with $b_2 = g_{i_2,2} - w_{i_2,2}$. Since $w_{i_2,2} > 0$ (otherwise we would have two binding signals) we must have $w_{i_2,1} = 0$ and signal i_2 must be critical in direction 2.

Now, assume that there is no critical signal in direction 1. Then, for all the signals, we must have either $b_1 < g_{i,1} - w_{i,1}$ or $w_{i,2} > 0$. Furthermore, since $b_1 = g_{j_1,1}$, then $w_{j_1,2} > 0$. We can build an augmentation circuit by using arcs $w_{i,2}$ when $w_{i,2} > 0$ and $w_{i,1}$ otherwise. We can have a flow change of $-\delta$ on all the arcs in direction 2 and $+\delta$ on all the arcs in direction 1. Since $w_{i_2,2}$ is reduced, b_2^* is increased by δ, which contradicts the fact that $\mathbf{w}^*$ is optimal. ∎

3.4 Subproblem Algorithm

As a matter of convenience, we will combine the arcs by pairs as follows

$$x_i = (w_{i,1} + b_1) - (w_{i,2} + b_2),$$

$$y_i = t_{i,1} + t_{i+1,2}$$

to obtain the condensed network of Figure 8. These new arcs will have the following bounds:

$$-g_{i,2} \leq x_i \leq g_{i,1}$$

$$t_{i,1}^- + t_{i+1,2}^- \leq y_i \leq t_{i,1}^+ + t_{i+1,2}^+$$

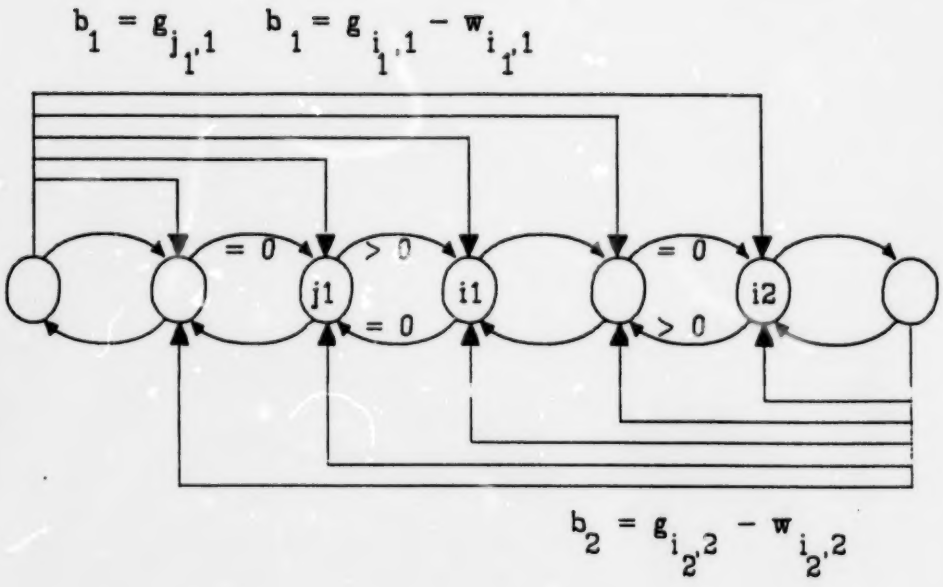
The inverse transformation is not unique, but we will use the following:

$$w_{i,1} = \max(0, +x_i), \tag{14}$$

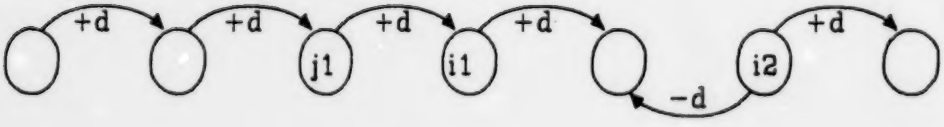
$$w_{i,2} = \max(0, -x_i), \tag{15}$$

$$t_{i,1} = t_{i,1}^- + (t_{i,1}^+ - t_{i,1}^-) \times (y_i - y_i^-)/(y_i^+ - y_i^-), \tag{16}$$

$$t_{i+1,2} = t_{i+1,2}^- + (t_{i+1,2}^+ - t_{i+1,2}^-) \times (y_i - y_i^-)/(y_i^+ - y_i^-), \tag{17}$$

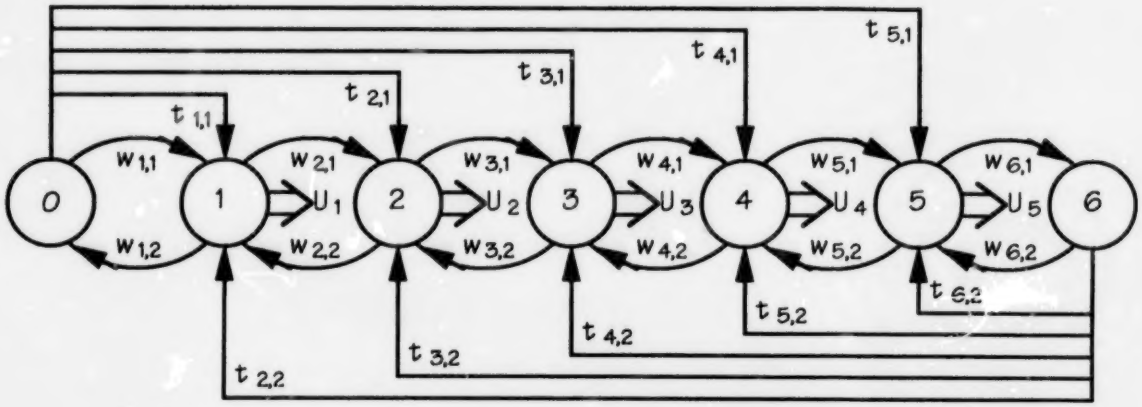


(a)

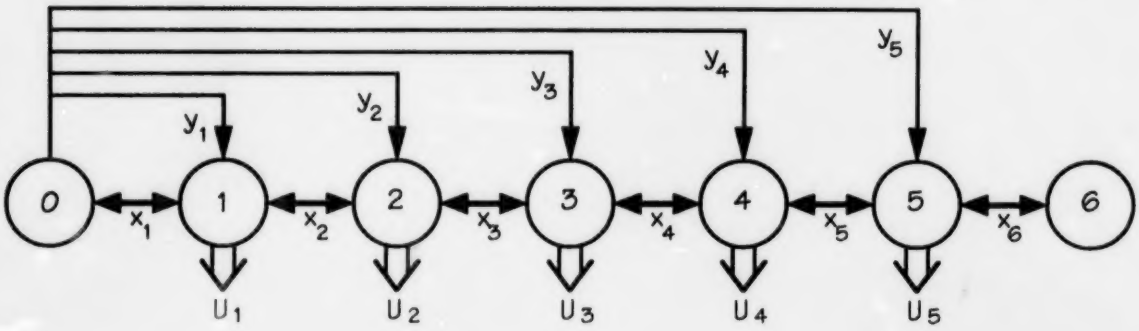


(b)

Figure 7: Proof of Property 5



(a)



(b)

Figure 8: Condensed Network

where

$$y_i^- = t_{i,1}^- + t_{i+1,2}^-$$

and

$$y_i^+ = t_{i,1}^+ + t_{i+1,2}^+.$$

Equations (14) and (15) have the effect of allocating all of x_i to either $w_{i,1}$ or $w_{i,2}$, depending on the sign of x_i, and will give us a solution satisfying property 2, while equations (16) and (17) allocate y_i to $t_{i,1}$ and $t_{i+1,2}$ according to its relative distance between its lower and upper bounds.

The algorithm we present to solve (SP) is recursive: given an optimal bandwidth for the first i signals, it will find the optimal bandwidth for the first $i+1$ signals by first assigning the smallest values it can to $w_{i+1,1}$ and $w_{i+1,2}$ and then making sure that property 1 is satisfied. If it is, then there is nothing else to do. If it is not, then the algorithm tries to find an augmentation circuit to reduce the value of $w_{i+1,1}$ or $w_{i+1,2}$.

Algorithm SP:

1. Initialization
   - $x_1 = 0$.
2. Signal by signal: for $i = 1$ to $n-1$, do:
   - $y_i = y_i^-$
   - $x_{i+1} = x_i + y_i - U_i$
   - $b_1 = g_{i+1,1} - x_i$
   - $b_2 = g_{i+1,2} + x_i$
   - Update flow: for $j = i-1$ down to 1, do:
     - $\delta = (b_1 - b_2)/2$
     - if $\delta \geq 0$ then $\delta = \min(\delta, y_j^- - y_j)$
       else $\delta = \max(\delta, y_j^- - y_j)$
     - if $\delta \neq 0$ then (update flow on path $0, j \ldots i+1$)
       - $b_1 = b_1 - \delta$
       - $b_2 = b_2 + \delta$
       - $y_j = y_j + \delta$
       - $x_k = x_k + \delta$ for $k = j+1 \ldots i$
     - $b_1 = \min(b_1, g_{j,1} - x_j)$
     - $b_2 = \min(b_2, g_{j,2} + x_j)$
   - Final update for signal i
     - $\delta = (b_1 - b_2)/2$
     - if $\delta \neq 0$ then $x_k = x_k + \delta$ for $k = 1 \ldots i$

3. Find the binding and critical signals:

   - let j_1 be such that $b_1 = g_{j_1,1}$,
   - let j_2 be such that $b_2 = g_{j_2,2}$,
   - let i_1 be such that $b_1 + x_{i_1} = g_{i_1,1}$ and $w_{i_1,2} = 0$,
   - let i_2 be such that $b_2 - x_{i_2} = g_{i_2,2}$ and $w_{i_2,1} = 0$.

4 SUBPROBLEM DUAL VARIABLES

Once the optimal solution is found, we must find the corresponding optimal dual variables. In this section, we will show how to derive the optimal dual variables with only the knowledge of which are the binding and the critical signals. We will consider three different cases, depending on the number of binding signals.

4.1 Optimal dual variables

4.1.1 Two binding signals

In this case, both bandwidths are bound by the green time of signals j_1 and j_2. The bandwidths can change only if the corresponding green times change. Any marginal change in the travel time bounds will have no effect on the bandwidths. A marginal change in the green time will produce at least a change of α in the objective function value; the effect will be bigger if the bandwidth in one direction equals the overall bandwidth.

If $b_1^* < b_2^*$ then a marginal change in $g_{j_1,1}$ will cause the same change in b^* and b_1^*. A change in $g_{j_2,2}$ will only affect b_2^*. The dual variables will be:

$$q_{j_1,1}^* = 1 + \alpha, \quad q_{j_2,2}^* = \alpha, \quad e_1^* = 1 \quad \text{and} \quad e_2^* = 0.$$

If $b_1^* > b_2^*$ then, for the same reason as above, the dual variables will be:

$$q_{j_1,1}^* = \alpha, \quad q_{j_2,2}^* = 1 + \alpha, \quad e_1^* = 0 \quad \text{and} \quad e_2^* = 1.$$

If $b_1^* = b_2^*$ then a marginal change in either $g_{j_1,1}$ or $g_{j_2,2}$ may affect b^* and we could choose either set of dual variables proposed above. In fact, any convex linear combination of the two preceding cases will do. We will use the following dual variables to be consistent with the next case:

$$q_{j_1,1}^* = 1/2 + \alpha, \quad q_{j_2,2}^* = 1/2 + \alpha, \quad e_1^* = 1/2 \quad \text{and} \quad e_2^* = 1/2.$$

4.1.2 No binding signal

From property 4, we know that $b_1^* = b_2^*$ and, consequently, any marginal change that affects either b_1^* or b_2^* can be followed by a flow change on an augmentation circuit between i_1 and i_2 that balances both bandwidths again. Thus, any marginal change will have a net effect of one half its value on the optimal bandwidth b^*.

If $g_{i_1,1}$ changes by δ, then b_1^*, b_2^* and b^* will all change by $\delta/2$, causing a change of $\delta/2 + \alpha(\delta/2 + \delta/2) = (1/2 + \alpha)\delta$ in the objective function value. We get the following dual variables:

$$q_{i_1,1}^* = 1/2 + \alpha, \quad q_{i_2,2}^* = 1/2 + \alpha, \quad e_1^* = 1/2 \quad \text{and} \quad e_2^* = 1/2.$$

The values of the other dual variables depend on the relative position of the two critical signals i_1 and i_2, according to property 3:

If $i_1 > i_2$, then

$$\left.\begin{array}{lcl} p_i & = & +(1/2+\alpha) \\ u_{i,1}^- & = & (1/2+\alpha) \\ u_{i+1,2}^- & = & (1/2+\alpha) \end{array}\right\} i_2 \leq i < i_1$$

If $i_1 < i_2$, then

$$\left.\begin{array}{lcl} p_i & = & -(1/2+\alpha) \\ u_{i,1}^+ & = & (1/2+\alpha) \\ u_{i+1,2}^+ & = & (1/2+\alpha) \end{array}\right\} i_1 \leq i < i_2$$

It is impossible to have $i_1 = i_2$ because, then, both signals would also be binding.

4.1.3 One binding signal

According to property 6, we know we also have two critical signals. There are two sub-cases, depending on whether the binding signal is j_1 or j_2, but we will examine only only the sub-case where the binding signal is j_1, the other one being similar.

If we increase the green time of signal j_1 by δ, we will increase the values of b_1^* and $b*$ by the same amount, producing a change of $(1 + \alpha)\delta$ in the objective function value. But the increase of b_1^* will force us to decrease $w_{i_1,1}^*$ by by δ and, following the augmentation circuit between i_1 and i_2, we will also increase $w_{i_2,2}^*$ and decrease b_2^* by by δ , causing a change of $-\alpha\delta$ in the objective function value. The net change in the objective function value will be δ.

If we increase the green time of signal i_1 by δ, we will be able to increase $w_{i_1,1}^*$ by the same amount to maintain the equality $g_{i_1,1} - w_{i_1,1}^* = b_1^*$. After following the augmentation circuit between i_1 and i_2, we will decrease $w_{i_2,2}^*$ by δ, thereby increasing b_2^* and causing a change of $\alpha\delta$ in the objective function value.

Changing the green time of signal i_2 will have an immediate effect on b_2^* and will cause a change of $\alpha\delta$ in the objective function value.

The dual variables for this sub-case are:

$$q_{j_1,1}^* = 1, q_{i_1,1}^* = \alpha, q_{i_2,2}^* = \alpha, e_1^* = 1 \text{ and } e_2^* = 0.$$

If $i_1 > i_2$, then

$$\left.\begin{array}{lcl} p_i & = & \alpha \\ u_{i,1}^- & = & \alpha \\ u_{i+1,2}^- & = & \alpha \end{array}\right\} i_2 \leq i < i_1$$

If $i_1 < i_2$, then

$$\left.\begin{array}{rcr} p_i & = & -\alpha \\ u^+_{i,1} & = & \alpha \\ u^+_{i+1,2} & = & \alpha \end{array}\right\} i_1 \le i < i_2$$

4.2 Feasibility

We must show that the proposed dual variables satisfy constraints (7)–(10) for each case and sub-case. For the sake of brevity, we will only show the proof for the case of two binding signals and $b^*_1 < b^*_2$, all the other cases being as straightforward.

$$\begin{array}{lrcl} (7) & e_1 + e_2 & = & 1 \\ & 1 + 0 & = & 1 \end{array}$$

$$\begin{array}{lrcll} (8) & \sum q_{i,d} - e_d & = & \alpha & d = 1,2, \\ & (1+\alpha) - 1 & = & \alpha & d = 1, \\ & \alpha - 0 & = & \alpha & d = 2, \end{array}$$

$$\begin{array}{lrcll} (9) & q_{i,1} + (p_i - p_{i-1}) & \ge & 0 & i = 1 \ldots n, \\ & 0 + (0 - 0) & = & 0 & i = 1 \ldots i_1 - 1, \\ & (1+\alpha) + (0 - 0) & = & (1+\alpha) & i = i_1, \\ & 0 + (0 - 0) & = & 0 & i = i_1 + 1 \ldots n, \\ (10) & q_{i,2} - (p_i - p_{i-1}) & \ge & 0 & i = 1 \ldots n, \\ & 0 - (0 - 0) & = & 0 & i = 1 \ldots i_2 - 1, \\ & \alpha - (0 - 0) & = & \alpha & i = i_2, \\ & 0 - (0 - 0) & = & 0 & i = i_2 + 1 \ldots n, \end{array}$$

$$\begin{array}{lrcrcll} (11) & u^+_{i,1} - u^-_{i,1} + p_i & \ge & 0 & & i = 1 \ldots n-1, \\ & 0 - 0 + 0 & = & 0 & & i = 1 \ldots n-1, \\ (12) & u^+_{i+1,2} - u^-_{i+1,2} + p_i & \ge & 0 & & i = 1 \ldots n-1, \\ & 0 - 0 + 0 & = & 0 & & i = 1 \ldots n-1 \end{array}$$

4.3 Optimality

We must show that the value of the objective function of problem (DSP) equals the optimal value of the objective function of problem (SP):

$$\sum g_{i,d} q_{i,d} + \sum U_i p_i + \sum (t^+_{i,d} u^+_{i,d} - t^-_{i,d} u^-_{i,d}) = b^* + \alpha(b^*_1 + b^*_2) \tag{18}$$

4.3.1 Two binding signals

In this case, the left-hand side of (18) reduces to

$$q_{i_1,1} b^*_1 + q_{i_2,2} b^*_2 \tag{19}$$

If $b^*_1 < b^*_2$ then $b^* = b^*_1$ and (19) becomes $(1 + \alpha)b^*_1 + \alpha b^*_2$, which equals the right-hand side of (18).

If $b_1^* > b_2^*$ Then $b^* = b_2^*$ and (19) becomes $\alpha b_1^* + (1+\alpha)b_2^*$, which is equal to the right-hand side of (18).

Finally, if$b_1^* = b_2^*$, then $b^* = b_1^* = b_2^*$ and (19) becomes $(1/2+\alpha)b_1^* + (1/2+\alpha)b_2^*$, which is equal to the right-hand side of (18).

4.3.2 No binding signal

We will develop the proof only for the first sub-case ($i_1 > i_2$) since the other sub-case is similar. First, we remove all the terms of (18) that are equal to zero, we factor out $(1/2+\alpha)$ and we get

$$\left(\frac{1}{2}+\alpha\right) \times \left(g_{i_1,1} + g_{i_2,2} + \sum_{i=i_2}^{i_1-1}(U_i - (t_{i,1}^- + t_{i+1,2}^-))\right)$$

Next, we use property 3 and equation (6) to get

$$\left(\frac{1}{2}+\alpha\right) \times \left(g_{i_1,1} + g_{i_2,2} + \sum_{i=i_2}^{i_1-1}((w_{i,1}^* + w_{i+1,2}^*) - (w_{i+1,1}^* + w_{i,2}^*))\right)$$

Now, we cancel out all the middle terms of the summation and we use the fact that $w_{i_1,2}^* = 0$ and $w_{i_2,1}^* = 0$ and we get

$$(1/2+\alpha)\left(g_{i_1,1} - w_{i_1,1}^* + g_{i_2,2} - w_{i_2,2}^*\right)$$

Finally, we use the definition of i_1 and i_2 to get

$$(1/2+\alpha)(b_1^* + b_2^*)$$

which is equal to $b^* + \alpha(b_1^* + b_2^*)$ since, according to property 4, $b_1^* = b_2^* = b^*$.

4.3.3 One binding signal

Once again, we will develop the proof only for the first sub-case ($i_1 > i_2$) since the other sub-case is similar. First, we remove all the terms of (18) that are equal to zero, we factor out α and we get

$$g_{j_1,1} + \alpha\left(g_{i_1,1} + g_{i_2,2} + \sum_{i=i_2}^{i_1-1}(U_i - t_{i,1}^- + t_{i+1,2}^-)\right)$$

Next, we use property 3 and equation (6) to get

$$g_{j_1,1} + \alpha\left(g_{i_1,1} + g_{i_2,2} + \sum_{i=i_2}^{i_1-1}((w_{i,1}^* + w_{i+1,2}^*) - (w_{i+1,1}^* + w_{i,2}^*))\right)$$

Now, we cancel out all the middle terms of the summation and we use the fact that $w_{i_1,2}^* = 0$ and $w_{i_2,1}^* = 0$ and we obtain

$$g_{j_1,1} + \alpha(g_{i_1,1} - w_{i_1,1}^* + g_{i_2,2} - w_{i_2,2}^*).$$

Finally, we use the definition of j_1, i_1 and i_2 to get $b^* + \alpha(b_1^* + b_2^*)$.

5 COMPUTATIONAL RESULTS

In our computational results we compared the effectiveness of the Benders decomposition algorithm to the traditional approach of solving problem (DSAVS) with an ordinary branch and bound algorithm. The two algorithms were programmed in FORTRAN and were compiled with the code optimization option of the DEC VAX compiler. The branch and bound approach used the mathematical programming routines of Land and Powell[1973], which are the same routines used by MAXBAND (Little et al.[1981]) and MAXBAND-86 (Messer et al.[1986]). The Benders decomposition approach also used the Land and Powell routines to solve the Master problem, thus eliminating the effect of the programmer's quality from the comparisons. The subproblem was solved with the specialized algorithm of section 3.4 and never took more than 1% of the total computing time. The tests were run on a VAX 11/785 and the CPU times do not include the input at the beginning, the output at the end of the program and the operating system overhead.

With the Benders decomposition algorithm, we also tested some technical variations that do not affect the problem formulation, but affect the computational efficiency.

We first consider the variable z, introduced in the master problem. The optimal value of z will equal the optimal bandwidth and, consequently will represent a percentage of the cycle time. As such, we can immediately deduce that its optimal value will be less than or equal to 1. This leaves us with the option of either specifying an upper bound for z or not. What we hope to gain from specifying an upper bound is a reduction of the solution time required to solve the master problem in the first few iterations of the algorithm because there are few constraints and the optimal value of z^K may be greater than 1. By not specifying an upper bound for the value of z, we must search the entire branch and bound tree to find an optimal solution that may have a value greater than 1. By specifying an upper bound of 1, we can terminate the search of the tree as soon as we find a solution with a value of z equal to 1. After a few iterations, we obtain a sufficient number of cuts and the optimal value of z^K falls below 1, then we perform the usual complete search of the tree. Going a bit further with the same idea, we can easily deduce that the optimal bandwidth cannot be greater than the smallest of all the green times and, consequently, use that number as a bound.

The other technical variation we considered has to do with the numerical accuracy of the routines used to solve the master problem. In their book [Land and Powell, 1973, p.182], the authors mention that the value of the non-zero coefficients of the A matrix should be within one or two orders of magnitude of each other and that scaling of some variables might be in order. The coefficients of z is always 1 and the coefficients of the m_i's are either 0, 1/2 or 1 and the coefficients of s can typically be as high as 250. We followed Land and Powell's suggestion and we tried to scale the value of s so its coefficients would be less than 100.

Combining these two modifications, we obtain a total of six variations to the Benders decomposition algorithm. Table 2 presents the computational results of a variety of problems solved with the seven different algorithms. The problems all came from a test problem used by Little et al.[1981]: the 10 signal Wisconsin Avenue problem. The problems with less than 10 signals were taken from the first signals, and the problems with more than 10 signals were generated by repeating the first ten signals.

The results indicate that most of the time all the variations of the Benders algorithm are faster than regular branch and bound. If we select the best of the six Benders variations, it

	B&B	Benders variants											
		s not scaled						s scaled					
		$z \leq \infty$		$z \leq$ Min $g_{i,d}$		$z \leq 1$		$z \leq \infty$		$z \leq$ Min $g_{i,d}$		$z \leq 1$	
N	CPU	CPU	K	CPU	K	CPU	K	CPU	K	CPU	K	CPU	K
5	1.70	1.90	14	0.17	5	1.24	12	2.29	15	0.07	3	1.26	12
6	7.27	1.27	11	1.04	12	0.96	10	0.96	10	0.86	10	0.94	10
7	12.08	1.61	11	2.75	17	1.56	11	2.49	13	1.92	14	2.54	13
8	23.60	2.20	10	2.44	16	3.55	14	3.54	14	2.65	18	4.43	15
9	19.71	19.23	19	6.05	21	6.33	16	21.40	22	5.28	18	8.03	16
10	55.25	10.89	16	18.61	23	7.49	16	14.83	18	9.92	20	11.06	19
11	65.88	27.35	22	18.67	26	24.55	24	17.98	20	6.67	21	17.93	24
12	120.55	35.72	23	17.42	26	7.62	18	41.05	23	10.05	23	7.58	18
13	175.97	251.45	21	34.35	31	128.79	23	230.94	28	52.23	28	75.32	22
14	208.40	268.67	31	39.91	35	97.54	30	326.18	27	25.53	32	54.34	23
15	246.22	475.06	30	133.71	46	109.48	37	127.79	31	57.72	40	192.17	29
16	289.42	406.80	32	226.56	43	184.69	33	560.98	27	74.42	39	365.49	35
17	563.02	512.45	31	138.88	52	875.35	43	488.32	32	179.60	53	128.12	30
18	918.79	991.13	50	374.32	58	645.88	39	950.00	34	246.36	53	399.78	35
19	587.29	852.64	34	871.65	58	823.24	49	3,544.67	51	248.04	59	545.63	36
20	1,256.03	1,714.69	42	1,083.47	72	1,001.16	44	2,720.07	49	308.17	60	2,184.97	45

Table 2: Computing results: B&B vs 6 Benders variants

is about ten times faster than MAXBAND.

Comparing the six Benders variations, we note that providing an upper bound for z gives much better results and that using an upper bound equal to the smallest green time gives more consistent results, in the sense that it is often the best or a close second. We see that scaling s does not provide any clear advantage for the smaller problem sizes, but can reduce the computing time by a factor of 3 for the larger problems.

The reason the results of the six Benders variations differ when the theoretical problem formulation is the same is because, for a given iteration of the algorithm, the technical differences may lead us to choose a different solution when the problem has more than one optimal solution. From that point on, the master problems differ and the subsequent Benders iterations differ.

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